The Bogolyubov–Krylov method of accelerated convergence in nonlinear mechanics

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A review is presented of the basic ideas in the studies of Bogolyubov and Krylov in which they proposed and substantiated the method of accelerated convergence in nonlinear mechanics and considered many important applications of this method based on successive changes of variables.

In 1934, Krylov and Bogolyubov proposed one of the numerous methods of the nonlinear mechanics that they constructed, the so-called asymptotic methods of nonlinear mechanics. This was the special method of successive changes of variables; in many cases, it is an effective tool for solving numerous interesting and important problems of nonlinear mechanics. In particular, this method was used to solve the important problem of the existence of a quasiperiodic regime with two fundamental frequencies in nonlinear oscillatory systems.

We shall consider the basic features of this method.[9] We consider a system of differential equations in the standard form[1)]

$$\frac{dx}{dt}=\varepsilon X(t,x,\varepsilon), \tag{1}$$

where $x=(x_1,...,x_n)$, $X=(X_1,...,X_n)$ are points of the n-dimensional Euclidean space E_n, t is the time, and ε is a small positive parameter.

In accordance with the general methods of nonlinear mechanics, we form for Eq. (1) the mth approximation:

$$x^{(m)}=\xi+\varepsilon F^{(1)}(t,\xi)+...+F^{(m)}(t,\xi), \tag{2}$$

in which the new variables ξ are solutions of the equation

$$\frac{d\xi}{dt}=\varepsilon\mathscr{P}^{(1)}(\xi)+...+\varepsilon^{(m)}\mathscr{P}^{(m)}(\xi). \tag{3}$$

Here, the functions $F^{(1)}(t,\xi),...,F^{(m)}(t,\xi)$ and $\mathscr{P}^{(1)}(\xi),...,\mathscr{P}^{(m)}(\xi)$ are chosen [on the basis of the known expressions for $X(t,\xi,\varepsilon)$] in such a way that the series (2) satisfy Eqs. (1) up to terms of order ε^{m+1}, provided that ξ is determined from Eqs. (3).

If now, having determined the functions $F^{(1)}(t,\xi),...,F^{(m)}(t,\xi)$, we regard the expression (2) not, as is usual in nonlinear mechanics, as an approximate asymptotic solution of the system (1), but as a certain change of variables, which transforms the unknown x to the new unknown ξ, then Eq. (1) is reduced to the equation

$$\frac{d\xi}{dt}=\varepsilon\mathscr{P}^{(1)}(\xi)+\varepsilon^2\mathscr{P}^{(2)}(\xi)+...+\varepsilon^m\mathscr{P}^{(m)}(\xi)+\varepsilon^{m+1}\mathscr{R}(t,\xi,\varepsilon), \tag{4}$$

which consists, so to speak, of an "integrable" part and a perturbation $\varepsilon^{m+1}\mathscr{R}(t,\xi,\varepsilon)$, which is a quantity of order ε^{m+1} and depends on the time t. At the same time, if the variable ξ satisfies Eq. (4), then the expression (2) is an exact solution of Eq. (1).

In the expressions (2)–(4), we now let m go to infinity. If the series (2) are convergent, then the system of equations (1) reduces to the "integrable" system

$$\frac{d\xi}{dt}=\varepsilon\mathscr{P}(\xi,\varepsilon), \tag{5}$$

where

$$\mathscr{P}(\xi,\varepsilon)=\lim_{m\to\infty}(\mathscr{P}^{(1)}(\xi)+\varepsilon\mathscr{P}^{(2)}(\xi)+...\varepsilon^m\mathscr{P}^{(m)}(\xi)).$$

However, in the general case such a development of the method proved to be impossible—already for systems with functions $X(t,x,\varepsilon)$ quasiperiodic in t, small divisors appear in the expressions (2), and the series (2) diverge. Because of this, the idea of reduction to the "integrable" system (5) remained impossible, and the method did not allow one to draw conclusions about the behavior of the solutions of the system (1).

However, for m finite and ε small, the series (2) can be regarded as approximate asymptotic solutions.

In 1963, following the publication of papers of Kolmogorov[6–8] and Arnol'd,[1–3] Nikolaĭ Nikolaevich Bogolyubov[4] developed a new method of successive changes of variables, and the idea of reducing the system (1) to the "integrable" form (5) found its real implementation, not only in the proof of a number of interesting and important theorems by Bogolyubov himself but also in the studies of various authors.

We give the gist of the idea of the cited studies of Kolmogorov and Arnol'd as applied to a conservative dynamical system determined by the canonical equations

$$\frac{dp}{dt}=-\frac{\partial H}{\partial q},\quad \frac{dq}{dt}=\frac{\partial H}{\partial p},\quad p=(p_1,...,p_n),$$

$$q=(q_1,...,q_n), \tag{6}$$

with an analytic Hamiltonian function $H(p,q,\varepsilon)$ that is 2π-periodic in q.

We assume that the Hamiltonian $H(p,q,\varepsilon)$ has the form

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$$H(p,q,\varepsilon)=H_0(p)+\varepsilon H_1(p,q)+\varepsilon^2\ldots, \tag{7}$$

i.e., the system (6) differs from an integrable system by a small perturbation.

Substituting the value (7) of $H(p,q,\varepsilon)$ in Eqs. (6), we obtain the system

$$\frac{dp}{dt}=-\varepsilon\frac{\partial H_1}{\partial q}+\ldots$$

$$\frac{dq}{dt}=\omega(p)+\varepsilon\frac{\partial H}{\partial p}+\ldots$$

$$\left(\omega(p)=\frac{\partial H_0}{\partial p}\right). \tag{8}$$

In the system of equations (8), we now make a canonical transformation in accordance with

$$p=p'+\varepsilon\frac{\partial S(p',q)}{\partial q}$$

$$q'=q+\varepsilon\frac{\partial S(p',q)}{\partial p'}, \tag{9}$$

and this reduces $H(p,q,\varepsilon)$ to the form

$$H(p,q,\varepsilon)=H_0'(p',\varepsilon)+\varepsilon^2 H_1'(p',q')+\ldots, \tag{10}$$

and Eqs. (8) to the form

$$\frac{dp'}{dt}=-\varepsilon^2\frac{\partial H_1'}{\partial q'}+\ldots$$

$$\frac{dq'}{dt}=\omega'(p')+\varepsilon^2\frac{\partial H_1'}{\partial p'}+\ldots. \tag{11}$$

Making in the system of equations (11) a transformation of the same type as (9), we obtain the equations

$$\frac{dp'}{dt}=-\varepsilon^4\frac{\partial H_1''}{\partial q''}+\ldots$$

$$\frac{dq''}{dt}=\omega''(p'')+\varepsilon^4\frac{\partial H_1''}{\partial p''}+\ldots, \tag{12}$$

etc., in which the orders of the "nonintegrable" corrections for the equations will be, respectively, proportional to $\varepsilon^2,\varepsilon^4,\varepsilon^8,\ldots\varepsilon^{2^s},\ldots$.

Whereas in the approximation process made by means of the expressions (2) we augment the right-hand sides with terms of higher order in order to raise the accuracy, in the process as given by Kolmogorov and Arnol'd there is a new element, which is that one and the same transformation is applied repeatedly. The resulting "accelerated convergence" of the process suppresses the influence of the small denominators that appear in the expressions of the change of variables (9), and for the "majority" of initial values of p the superposition of such substitutions converges.

In the solution of some problems of nonlinear mechanics, it is possible to establish for the corresponding differential equations the existence of integral (invariant) manifolds, which possess the property of asymptotic attraction of neighboring trajectories.

For example, suppose that the dynamical system is characterized by the equations

$$\frac{dx}{dt}=X(x,\varepsilon), \tag{13}$$

where $x=(x_1,\ldots,x_n)$, $X=(X_1,\ldots,X_n)$ are the vectors of an n-dimensional Euclidean space, and ε is a small positive parameter.

Under certain conditions, it is possible to establish for the system of equations (13) the existence of an invariant toroidal manifold:

$$x=\Phi(\varphi),\quad \varphi=(\varphi_1,\ldots,\varphi_m). \tag{14}$$

In this case, the original system (13) reduces to an equation on a torus,

$$\frac{d\varphi}{dt}=\nu+f(\varphi,\varepsilon), \tag{15}$$

where $\varphi=(\varphi_1,\ldots,\varphi_m)$, $\nu=(\nu_1,\ldots,\nu_m)$, and $f=(f_1,\ldots,f_m)$ is a periodic function of φ, which may be small [by virtue of the presence of ε on the right-hand side of Eq. (13)], i.e., the variation of φ is close to uniform rotation with constant angular velocity ν.

Under certain conditions, the manifold (14) possesses the property of asymptotic attraction of the trajectories of all solutions of Eqs. (13) that do not lie on the torus (14).

It is of interest not only to find the integral manifold (14) but also to investigate the behavior of the integral curves that lie on this manifold.

The investigations of Krylov and Bogolyubov in 1934 that were devoted to this question and were based on the method of successive changes of variable by means of the transformations (2) were made using the results of Poincaré and Denjoy on mappings of the circle onto itself.

However, their theory relates to the one-dimensional case, when the original system of differential equations (13) reduces to two equations:

$$\frac{d\varphi}{dt}=\nu+f(\varphi,\theta)$$

$$\frac{d\theta}{dt}=\omega. \tag{16}$$

In accordance with the Poincaré–Denjoy theory, the behavior of the solutions on the two-dimensional torus (16) is characterized by the number of revolutions Ω: 1) if Ω is irrational, then the solutions on the torus are quasiperiodic; 2) if Ω is rational, then there are periodic solutions, and all the remaining solutions approach them with the passage of time.

If the original system of differential equations (13) is reduced to the form (16), then, as was shown by Krylov and Bogolyubov[10] in 1934, one can prove the existence of a quasiperiodic solution with two fundamental frequencies ω_1 and ω_2 and establish its stability.

At that time, it did not prove possible to study the general case in the framework of the Poincaré–Denjoy theory.

In 1963 Bogolyubov succeeded, by combining the method of accelerated convergence with the method of integral manifolds that he had developed, taking into account at the same time some specific features inherent in nonlinear oscillatory systems, in significantly extending the domain of applicability of the method of successive changes of variables and in solving the problem of the existence of quasiperiodic solutions for the general case $n>2$.

We turn to the formulation of the basic problem considered by Bogolyubov.[4]

In the investigation of the system of equations (13), it is usually convenient to replace the variables $x=(x_1,...,x_n)$ by new variables h, φ in such a way that the original equations (13) are reduced to a system of the form

$$\frac{dh}{dt}=Hh+F(h,\varphi)$$

$$\frac{d\varphi}{dt}=\nu+f(h,\varphi), \tag{17}$$

where $h=(h_1,...,h_n)$, $\varphi = (\varphi_1,...,\varphi_{n_0})$ (the sum $n+n_0$ of the dimensions of the vectors h and φ is equal to the dimension of the vector x and is denoted $n+n_0$ for convenience); at the same time, we assume that the real parts of the eigenvalues of the $n\times n$ matrix H are all negative:

$$|e^{Ht}|\leqslant\mathscr{P}e^{-\alpha t},\quad t\geqslant 0, \tag{18}$$

where $\mathscr{P}>0$, $\alpha>0$ are constants; the vector functions $F(h,\varphi)$ and $f(h,\varphi)$ are small for sufficiently small h and are regular.

Even if the functions on the right-hand sides of the system of equations (17) are analytic and arbitrarily small, one cannot directly prove the existence of an analytic torus in the complex domain for these equations. This is made obvious by a simple example.

Consider the system of differential equations

$$\frac{dh}{dt}=-\alpha h+f(\varphi)=-\alpha h+\sum_{(k)}\rho^{(k)}e^{i(k,\varphi)}$$

$$\frac{d\varphi}{dt}=\nu+\varepsilon\gamma\quad(h=h_1,\ \varphi=(\varphi_1,\varphi_2)). \tag{19}$$

For the system (19), we readily find the invariant manifold

$$h=S(\varphi), \tag{20}$$

where

$$S(\varphi)=\sum_{(k)}\frac{e^{i(k,\varphi)}}{ik(\nu+\varepsilon\gamma)+\alpha}\quad(k=(k_1,k_2)), \tag{21}$$

and it is obvious that for real ε and γ the torus (21) always exists. However, if ε or γ are complex, then we can always adjust them in such a way that

$$k(\nu+\mathrm{Re}\,\varepsilon\gamma)=0\ \text{ and }\ k(\mathrm{Im}\,\varepsilon\gamma)+\alpha=0.$$

In this case, we can obtain for the function $S(\varphi)$ a set of poles for arbitrarily small ε, and, therefore, we cannot directly establish the existence of an invariant torus that is analytic with respect to both the angle φ and the parameter ε in the neighborhood of the value $\varepsilon=0$.

Note also that since in the construction of the change of variables (2) sums of the type $(m\omega)$ occur in the denominators, we cannot expand ω in powers of a small parameter, and therefore it is expedient to express the "frequencies of the zeroth approximation" ν in Eqs. (17) in terms of the exact frequencies ω and not find ω from ν and f; rather, it is better to assume that ω are given and to determine $\Delta=\nu-\omega$ as a function of ω.

Substituting $\nu=\omega+\Delta$ in Eqs. (17), we obtain the system

$$\frac{dh}{dt}=Hh+F(h,\varphi,\Delta)$$

$$\frac{d\varphi}{dt}=\omega+\Delta+f(h,\varphi,\Delta). \tag{22}$$

We now assume that for the system of equations (22) the following conditions hold: $F(h,\varphi,\Delta)$ and $f(h,\varphi,\Delta)$ are analytic functions of the complex variables h, φ, and Δ in the region

$$\|h\|\leqslant\eta_1,\quad|\mathrm{Im}\,\varphi|\leqslant\rho,\quad|\Delta|\leqslant\sigma_1, \tag{23}$$

that are sufficiently small for sufficiently small h, Im φ, and Δ, and in the region (23) they satisfy the conditions

$$\|F(h,\varphi,\Delta)\|\leqslant N,\quad n\left\|\frac{\partial F(h,\varphi,\Delta)}{\partial h_q}\right\|\leqslant L,$$

$$|f(h,\varphi,\Delta)|\leqslant M, \tag{24}$$

where the constants N, L, M, η, ρ_1, σ_1 satisfy certain relations that we do not give here and are used in the process of the complicated proof of the theorem.

In addition, we have introduced the norm

$$\|h\|=\sup_{\substack{k=1,...,n\\ 0\leqslant t<\infty}}|e^{Ht}|e^{\alpha t}, \tag{25}$$

where H, the square n-dimensional matrix in Eq. (22), satisfies the condition

$$|e^{Ht}|\leqslant\mathscr{P}e^{-\alpha t}\quad\text{for } t\geqslant 0,\quad\alpha>0,\quad\mathscr{P}=\text{const}\geqslant 1. \tag{26}$$

Suppose, in addition, that the real $\omega=(\omega_1,\omega_2,...,\omega_{n_0})$ satisfy the condition

$$|(m,\omega)|\geqslant k|m|^{-(n_0+1)}, \tag{27}$$

where n_0 is the dimension of the space of ω, and $|m|=|m_1|+|m_2|+...+|m_{n_0}|$, $m_1,m_2,...,m_{n_0}$ are arbitrary integer (positive and negative) numbers.

The condition (27) is necessary to ensure that in the construction of the changes of variables we do not encounter denominators of the type (m,ω).

However, it is well known that if one considers the sphere in the space $\omega = (\omega_1,\omega_2,...,\omega_{n_0})$ then the relative measure of the set of the ω for which the condition (27) is not satisfied tends to zero together with k. Thus, for sufficiently small k "most" ω satisfy the inequality (27).

Thus, combining the method of accelerated convergence with the method of integral manifolds, Bogolyubov proved the following fundamental theorem, in which he took into account in sophisticated arguments the properties of nonlinear systems.

Theorem (Bogolyubov's Theorem). If in the system of equations (22) the functions $F(h,\varphi,\Delta)$ and $f(h,\varphi,\Delta)$ satisfy all the conditions given above, then for an appropriate choice of $\Delta=D^{(\infty)}$ these equations have a quasiperiodic solution with frequencies $\omega = (\omega_1,\omega_2,...,\omega_{n_0})$ of the form

$$\begin{aligned} &h_t=S^{(\infty)}(\omega t+\vartheta_0) \\ &\varphi_t=\omega t+\vartheta_0+\Phi^{(\infty)}(S^{(\infty)}(\omega t+\vartheta),\omega t+\vartheta_0,0). \end{aligned} \tag{28}$$

If h_t and φ_t are any solutions of the system (22) whose initial values h_0 and φ_0 satisfy the condition

$$\|h_0\|\leqslant\frac{\eta_1}{2},\quad |\mathrm{Im}\,\varphi_0|\leqslant\frac{\rho_1}{2}\left(1-\frac{1}{2n}\right),$$

then h_t and φ_t will converge asymptotically to this quasiperiodic solution. If in addition to the above conditions the functions $F(h,\varphi,\Delta)$ and $f(h,\varphi,\Delta)$ are analytic functions of the parameter ε in the region $0\leqslant\varepsilon\leqslant\varepsilon_0$, then $D^{(\infty)}$, $\Phi^{(\infty)}$, and $S^{(\infty)}$ will also be analytic functions of ε in the region $0\leqslant\varepsilon\leqslant\varepsilon_0$.

We shall not dwell on the detailed proof of this theorem; we merely note that an important part of the proof is the construction of transformations

$$\begin{aligned} &\varphi=\vartheta+\Phi^{(\infty)}(h,\vartheta,0), \\ &\Delta=D^{(\infty)}, \end{aligned} \tag{29}$$

by means of which the system of equations (22) is reduced to the form

$$\begin{aligned} &\frac{dh}{dt}=Hh+F(h,\vartheta,\Delta). \\ &\frac{d\vartheta}{dt}=\omega. \end{aligned} \tag{30}$$

In accordance with the usual devices, an integral manifold $h=S^{(\infty)}(\omega t+\vartheta_0)$ is determined for the system of equations (30).

The construction of the transformation (29) uses a rapidly convergent iterative process of transformation that takes the following form. In the system, one introduces the change of variables

$$\begin{aligned} &\varphi=\varphi^{(1)}+u^{(1)}(h,\varphi^{(1)},\Delta^{(1)}), \\ &\Delta=\Delta(\Delta^{(1)}), \end{aligned} \tag{31}$$

as a result of which we obtain for h and $\varphi^{(1)}$ the equations

$$\frac{dh}{dt}=Hh+F_1(h,\varphi^{(1)},\Delta^{(1)}).$$

$$\frac{d\varphi^{(1)}}{dt}=\omega+\Delta^{(1)}+f_1(h,\varphi^{(1)},\Delta^{(1)}). \tag{32}$$

Applying to the system (32) the transformation

$$\begin{aligned} &\varphi^{(1)}=\varphi^{(2)}+u^{(2)}(h,\varphi^{(2)},\Delta^{(2)}), \\ &\Delta^{(1)}=\Delta^{(1)}(\Delta^{(2)}), \end{aligned} \tag{33}$$

we arrive at the equations

$$\begin{aligned} &\frac{dh}{dt}=Hh+F_2(h,\varphi^{(2)},\Delta^{(2)}) \\ &\frac{d\varphi^{(2)}}{dt}=\omega+\Delta^{(2)}+f_2(h,\varphi^{(2)},\Delta^{(2)}). \end{aligned} \tag{34}$$

In the sth step of this process, we make the transformation

$$\begin{aligned} &\varphi^{(s+1)}=\varphi^{(s)}+u^{(s)}(h,\varphi^{(s)},\Delta^{(s)}), \\ &\Delta^{(s-1)}=\Delta^{(s-1)}(\Delta^{(s)}) \end{aligned} \tag{35}$$

and we arrive at the equations

$$\begin{aligned} &\frac{dh}{dt}=Hh+F_s(h,\varphi^{(s)},\Delta^{(s)}), \\ &\frac{d\varphi^{(s)}}{dt}=\omega+\Delta^{(s)}+f_s(h,\varphi^{(s)},\Delta^{(s)}). \end{aligned} \tag{36}$$

In making the chain of transformations (31), (33),... (35),..., Bogolyubov performed scrupulous calculations and estimates that ensure at each step analyticity of the substitutions and, ultimately, accelerated convergence of the entire process proportional to $\varepsilon^2,\varepsilon^4,\varepsilon^8,...\varepsilon^{2^s}...$.

Subsequently, Bogolyubov's method of accelerated convergence ensured by successive changes of variables was used to construct the general solution of a nonlinear differential equation in the neighborhood of a quasiperiodic solution.

Considering, to simplify the calculations, the case when the n-dimensional square matrix H in the system (30) degenerates into a vector $\beta=(\beta_1,\beta_2,...,\beta_n)$ and assuming that all β have negative real parts, we arrive at the system

$$\begin{aligned} &\frac{dh}{dt}=\beta h+F(h,\varphi) \\ &\frac{d\varphi}{dt}=\omega. \end{aligned} \tag{37}$$

To construct solutions of the system (37), we apply the method of successive changes of variables; as a result, the β in (37) will be changed, tending in the limit to definite values, which we denote by $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$. They are the "true" coefficients of the linear system of differential equations that we obtain after transformation of the system (37). Therefore, as in the case with the frequencies ω, it is not expedient to find α from β and F but, assuming α is given, it is better to determine certain corrections $\xi=\beta-\alpha$ ($\xi=(\xi_1,\xi_2,...,\xi_n)$) as functions of α:

$$\xi=\xi(\alpha). \tag{38}$$

Thus, we introduce in the system (37) corrections ξ, after which it takes the form

$$\frac{dh}{dt}=(\alpha+\xi)h+F(h,\varphi,\xi)$$
$$\frac{d\varphi}{dt}=\omega, \tag{39}$$

where $F(h,\varphi,\xi)$ is an analytic function of the complex arguments h, φ, ξ in the region

$$|h|\leqslant\eta,\quad |\mathrm{Im}\,\varphi|\leqslant\rho,\quad |\xi|\leqslant\sigma. \tag{40}$$

Our task is to find a transformation

$$h=g+\vartheta^{(\infty)}(g,\varphi) \tag{41}$$

that is analytic in g and φ and a $\xi=\xi^{(\infty)}$ for which the system (39) is reduced to a system of linear differential equations with constant coefficients:

$$\frac{dg}{dt}=\alpha g$$
$$\frac{d\varphi}{dt}=\omega. \tag{42}$$

Then, integrating the system (42), we obtain the general solution of the system (39) in the form

$$h=Ce^{\alpha t}+\vartheta^{(\infty)}(Ce^{\alpha t},\omega t+\vartheta_0)$$
$$\varphi=\omega t+\vartheta_0, \tag{43}$$

which contains $n=n_0$ arbitrary constants $C=(C_1,C_2,...,C_n),\vartheta_0=(\vartheta_{01},\vartheta_{02},...,\vartheta_{0n_0})$.

Combining this solution with the quasiperiodic (28), we obtain the general solution of the system (27) in the neighborhood of the quasiperiodic solution (28).[2)]

As a result, the following theorem can be proved.

Theorem.[11] Suppose that for the system of equations

$$\frac{dh}{dt}=(\alpha=\xi)h+F(h,\varphi,\Delta,\xi)$$
$$\frac{d\varphi}{dt}=\omega+\Delta+f(h,\varphi,\Delta,\xi) \tag{44}$$

all the necessary conditions (see, for example, Ref. 11) are satisfied.

Then for an appropriate choice of $\Delta=D^{(\infty)}(0)$ and $\xi=\xi^{(\infty)}(0)$ the system (44) can be reduced by the change of variables

$$h=g+\vartheta^{(\infty)}(g,\vartheta,0)$$
$$\varphi=\vartheta+\Phi^{(\infty)}(g+\vartheta^{(\infty)}(g,\vartheta,0),\vartheta,0) \tag{45}$$

to a linear system with constant coefficients:

$$\frac{dg}{dt}=\alpha g$$
$$\frac{d\vartheta}{dt}=\omega, \tag{46}$$

and after integration of this system the general solution of the original system (44) will have the form

$$h_t=Ce^{\alpha t}+\vartheta^{(\infty)}(Ce^{\alpha t},\omega t+\vartheta_0,0),$$
$$\varphi_t=\omega t+\vartheta_0+\Phi^{(\infty)}(Ce^{\alpha t}+\vartheta^{(\infty)}(Ce^{\alpha t},\omega t+\vartheta_0,0),\omega t$$
$$+\vartheta_0,0), \tag{47}$$

where the $n+n_0$ arbitrary constants C, ϑ_0 belong to the region

$$|C|\leqslant\frac{\eta}{2},\quad |\mathrm{Im}\,\vartheta_0|\leqslant\frac{\rho}{2}. \tag{48}$$

With the passage of time, the solution (47) tends to the stationary quasiperiodic solution

$$h(\omega t)=\vartheta^{(\infty)}(0,\omega t+\vartheta_0,0)$$
$$\varphi(\omega t)=\omega t+\vartheta_0+\Phi^{(\infty)}(\vartheta^{(\infty)}(0,\omega t+\vartheta_0,0),\omega t+\vartheta_0,0), \tag{49}$$

where $|\mathrm{Im}\vartheta_0|\leqslant(\rho/2)(1-\frac{1}{2}n)$, in accordance with the law

$$|h_t-h(\omega t)|\leqslant\tfrac{3}{2}\,|C|e^{-\bar{\alpha}t}.$$
$$|\varphi_t-\varphi(\omega t)|\leqslant\tfrac{3}{2}\,C_1(r_0)\,|C|e^{-\bar{\alpha}t}. \tag{50}$$

This theorem on the possibility of reducing the system of equations (44) and, therefore, the system (1) to the equations (46) with constant coefficients and the construction of the general solution in the neighborhood of the stable stationary quasiperiodic solution (49) makes it possible to investigate the behavior of the solutions in the neighborhood of a quasiperiodic solution and opens up prospects for further investigation of other forms of equations containing a small parameter.

It is well known that systems of linear differential equations with quasiperiodic coefficients occupy an important place in the theory of differential equations. We shall therefore consider some results related to the study of systems of the form

$$\frac{dx}{dt}=Ax+\mathscr{P}(\varphi)x$$
$$\frac{d\varphi}{dt}=\omega, \tag{51}$$

where A is a constant, $\mathscr{P}(\varphi)$ are $n\times n$ real (for real φ) matrices that are 2π-periodic in $\varphi=(\varphi_1,\varphi_2,...,\varphi_m)$; $\omega=(\omega_1,\omega_2,...,\omega_m)$ are the frequencies of the matrix $\mathscr{P}(\omega t)$; $x=(x_1,x_2,...,x_n)$ is an n-dimensional vector; and t is the time.

For the system of equations (51), we try to find a change of variables

$$x=\Phi(\varphi)y \tag{52}$$

with nondegenerate periodic and real (for real φ) matrix $\Phi(\varphi)$ that reduces the system (51) to a system with constant coefficients

$$\frac{dy}{dt}=A_0 y, \quad \frac{d\varphi}{dt}=\omega, \tag{53}$$

where A_0 is a constant $n\times n$ real matrix.

This problem has been considered by many authors (see the references in Ref. 5). For the case of a linear system with periodic coefficients, i.e., the system (51) for $m=1$, the Floquet–Lyapunov results, which prove the existence of the substitution (52), are well known. The problem of the reducibility of a system with quasiperiodic coefficients has not yet been completely solved.

We now give some results obtained when one seeks solutions of the system (51) for the case in which the vector function $\mathscr{P}(\varphi)$ is small, using the construction of the reducing matrix $\Phi(\varphi)$ by means of Bogolyubov's method of accelerated convergence ensured by successive changes of variables. By means of this method, the system of equations (51) can be reduced to a system (53) with constant coefficients, using a reducing matrix $\Phi(\varphi)$ expressed by rapidly convergent series. The following theorem holds (for the proof, see Refs. 5 and 12).

Theorem. Suppose that the right-hand side of the system of equations

$$\begin{aligned}\frac{dx}{dt}&=Ax+\mathscr{P}(\varphi)x\\ \frac{d\varphi}{dt}&=\omega\end{aligned} \tag{54}$$

satisfies the following conditions.

1. The matrix $\mathscr{P}(\varphi)$ is 2π-periodic in $\varphi=(\varphi_1,\varphi_2,\ldots,\varphi_m)$, is analytic in the region

$$|\mathrm{Im}\,\varphi|=\sup_{\alpha}|\mathrm{Im}\,\varphi_\alpha|\leqslant\rho_0 \quad (\rho_0>0), \tag{55}$$

and is real for real φ.

2. For certain positive ε and d, the inequality

$$|(k,\omega)|\geqslant\varepsilon|k|^{-d} \quad (|k|\neq 0) \tag{56}$$

holds for all integer vectors $k=(k_1,k_2,\ldots,k_m)$.

3. The eigenvalues $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ of the matrix A have different real parts.

Then one can find a sufficiently small positive constant $\mathscr{M}_0$ such that for

$$|\mathscr{P}(\varphi)|=\sum_{i,j=1}^{n}|\mathscr{P}_{ij}(\varphi)|\leqslant\mathscr{M}_0 \tag{57}$$

the system of equations (54) can be reduced by means of a nondegenerate change of variables

$$x=\Phi(\varphi)y \tag{58}$$

with matrix $\Phi(\varphi)$ that is 2π-periodic in φ, analytic and analytically invertible in the region

$$|\mathrm{Im}\,\varphi|\leqslant\frac{\rho_0}{2}, \tag{59}$$

and real for $\mathrm{Im}\,\varphi=0$ to the form

$$\begin{aligned}\frac{dy}{dt}&=A_0 y\\ \frac{d\varphi}{dt}&=\omega,\end{aligned} \tag{60}$$

where A_0 is a constant matrix.

In accordance with this theorem, the fundamental matrix of solutions of the system (54) has the form

$$X=\Phi(\omega t+\varphi_0)e^{A_0 t}, \quad \varphi=\omega t+\varphi_0, \tag{61}$$

where the matrix $\Phi(\omega t+\varphi_0)$ is nondegenerate and quasiperiodic in t, real for real φ_0, and possesses frequency basis $\omega=(\omega_1,\omega_2,\ldots,\omega_m)$.

With allowance for Lagrange's well-known expression that represents an analytic function of the matrix $f(A)$ as a polynomial in A and the circumstance that by virtue of the smallness of the matrix $\mathscr{P}(\varphi)$ the eigenvalues $\lambda_1^0,\lambda_2^0,\ldots,\lambda_n^0$ of the matrix A_0 are close to the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ of the matrix A, i.e., also have different real parts, the fundamental matrix (61) can be represented in the form

$$X=\Phi(\omega t+\varphi_0)\sum_{k_0=1}^{n}\frac{(A_0-\lambda_1^0)\ldots(A_0-\lambda_{k_0-1}^0)(A_0-\lambda_{k_0+1}^0)\ldots(A_0-\lambda_n^0)}{(\lambda_{k_0}^0-\lambda_1^0)\ldots(\lambda_{k_0}^0-\lambda_{k_0-1}^0)(\lambda_{k_0}^0-\lambda_{k_0+1}^0)\ldots(\lambda_{k_0}^0-\lambda_n^0)}e^{\lambda_{k_0}^0 t}. \tag{62}$$

[1)] As is well known, the expression "standard form" is (in accordance with the terminology proposed by Krylov and Bogolyubov) used for differential equations whose right-hand side is proportional to a small parameter ε. Many problems of nonlinear mechanics containing a small parameter can be reduced to such equations.

[2)] The possibility of solution of such a problem was pointed out by Kolmogorov in Ref. 7, and some results close to those presented were obtained by Belaga (see the references in Ref. 5).

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